(1)

$$\sum_{k=n+1}^{3n}\frac{1}{k}(\log k-\log n)=\sum_{k=n+1}^{3n}\frac{1}{k}\log\frac{k}{n}$$

$$=\frac{1}{n+1}\log\frac{n+1}{n}+\frac{1}{n+2}\log\frac{n+2}{n}+\cdots\cdots+\frac{1}{n+n}\log\frac{n+n}{n}$$

$$+\frac{1}{2n+1}\log\frac{2n+1}{n}+\frac{1}{2n+2}\log\frac{2n+2}{n}+\cdots\cdots+\frac{1}{2n+n}\log\frac{2n+n}{n}$$

$$=\frac{1}{n}\left\{\frac{1}{1+\frac{1}{n}}\log\left(1+\frac{1}{n}\right)+\frac{1}{1+\frac{2}{n}}\log\left(1+\frac{2}{n}\right)\right.$$

$$\left.+\cdots\cdots+\frac{1}{1+\frac{n}{n}}\log\left(1+\frac{n}{n}\right)\right\}$$

$$+\frac{1}{n}\left\{\frac{1}{2+\frac{1}{n}}\log\left(2+\frac{1}{n}\right)+\frac{1}{2+\frac{2}{n}}\log\left(2+\frac{2}{n}\right)\right.$$

$$\left.+\cdots\cdots+\frac{1}{2+\frac{n}{n}}\log\left(2+\frac{n}{n}\right)\right\}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\log\left(1+\frac{k}{n}\right)+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2+\frac{k}{n}}\log\left(2+\frac{k}{n}\right)$$

よって，

$$\lim_{n\to\infty}\sum_{k=n+1}^{3n}\frac{1}{k}(\log k-\log n)=\int_0^1\frac{1}{1+x}\log(1+x)\,dx+\int_0^1\frac{1}{2+x}\log(2+x)\,dx$$

$$=\frac{1}{2}\Big[\{\log(1+x)\}^2+\{\log(2+x)\}^2\Big]_0^1$$

$$=\frac{1}{2}(\log 3)^2$$

(2) (1) と同様にして

$$\sum_{k=n+1}^{mn}\frac{1}{k}(\log k-\log n)=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\log\left(1+\frac{k}{n}\right)+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2+\frac{k}{n}}\log\left(2+\frac{k}{n}\right)$$

$$+\cdots\cdots+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{j+\frac{k}{n}}\log\left(j+\frac{k}{n}\right)$$

$$+\cdots\cdots+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{m-1+\frac{k}{n}}\log\left(m-1+\frac{k}{n}\right)$$

よって，

$$\lim_{n\to\infty}\sum_{k=n+1}^{mn}\frac{1}{k}\left(\log k-\log n\right)=\int_0^1\left\{\frac{1}{1+x}\log(1+x)+\frac{1}{2+x}\log(2+x)\right.$$

$$\left.+\cdots\cdots+\frac{1}{m-1+x}\log(m-1+x)\right\}dx$$

$$=\frac{1}{2}\left[\left\{\log\left(1+x\right)\right\}^2+\left\{\log\left(2+x\right)\right\}^2+\cdots+\left\{\log\left(m-1+x\right)\right\}^2\right]_0^1$$

$$=\frac{1}{2}\left[\left(\log 2\right)^2+\left(\log 3\right)^2+\cdots\cdots+\left(\log m\right)^2\right.$$

$$\left.-\left(\log 1\right)^2-\left(\log 2\right)^2-\cdots\cdots-\left\{\log\left(m-1\right)\right\}^2\right]$$

$$=\frac{1}{2}\left(\log m\right)^2$$